Synchronization in Gradient Networks

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Abstract

The contradiction between the fact that many empirical networks possess power-law degree distribution and the finding that network of heterogeneous degree distribution is difficult to synchronize has formed a paradox in the study of network synchronization. Surprisingly, we find that this paradox can be well solved when proper gradients are introduced to the network links, i.e. heterogeneous degree distribution is in favor of synchronization in gradient networks. We analyze the general properties of gradient networks and explore their functions in enhancing network sychronizability. Based on these understandings, we suppose the basic principles for constructing efficient gradient networks and propose a specific coupling scheme as verification. Comparing to the previous asymmetric coupling schemes, the new scheme not only possesses a much stronger synchronizability but also uses few network information. Moreover, under the framework of gradient network, the factors which had been employed in former studies in improving network synchronizability can be well unified and identified. The validity of our findings is verified by analytical estimates on the behavior of eigenvalues as well as directed simulations on coupled nonidentical oscillators. Our study therefore suggests that, in addition to the topology advantage, scale-free networks also manifest their dynamical advantage given proper gradients are considered.

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I. INTRODUCTION

The study of complex networks has attracted a great deal of interest since the discoveries of the small-world [1] and scale-free [2] properties in many natural and man-made networks [3, 4, 5]. While in initial studies the nodes and links of a network are treated as identical, recent studies have extended to the heterogeneous networks of scaled nodes [6] and weighted links [7] where many new features and properties are discovered. Meanwhile, in many practical systems the scalars, which usually reflect the different characteristics among the nodes, and the weights, which usually characterize the information transport capacities on the links, are closely correlated. Typical examples include the co-authorship networks [8], where the scalar can be regarded as the number of papers published by one researcher and the weight represents the collaboration times between two researchers, and the world-wide airline network (WWAN) [9], where the scalar can be the airport capacity and the weight represents the amount of transportation between two airports. In both cases the links which connect nodes of larger scalar often assume larger weights. However, the latter is different to the former in that in WWAN the transportation is directed and, in most cases, the mass flows in two directions are not balanced, i.e., there exists *gradient* on the links. The gradient networks, which are defined as *the directed graphs formed by local gradients of a scalar field distributed on the nodes*, are ubiquitous in nature and play an important role in many biological and technical systems [10, 11]. For example, it has been shown that the congestion tendency of traffic networks can be drastically reduced when gradient is considered [10, 11].

It is well known that the collective behavior of complex systems is strongly influenced by their underling coupling structures. One typical example is chaos synchronization in complex networks [5, 12, 13]. Compare to the regular networks, the synchronizability of both the small-world and scale-free networks (SFN) are drastically enhanced due to the decreased average distance [14]. However, as shown by the recent studies [15, 16, 17], the heterogeneous distribution of both the node degree and the link weight could suppress the synchronizability. The contradiction between the fact that many empirical networks have the property of heterogeneity and the finding that synchronizability are suppressed in heterogeneous networks has formed a paradox in the study of network synchronization. This paradox has stimulated the searching of optimal network configuration in SFN. Specifically, for a SFN of given topology and total coupling cost, people want to know how to distribute the couplings could efficiently promote the synchronizability [18, 19, 20, 21]. In Ref. [18] the authors proposed to distribute the incoming coupling strengths according to the

local information of node degree (hereafter we mark it as M-scheme). It is found that the synchronizability is solely determined by the average degree, independent of the degree distribution and the system size. In particular, under the condition of uniform coupling capacity distribution, SFN achieves its maximum synchronizability which is superior to network of homogeneous degree distribution. In Ref. [20] the incoming coupling strengths are proposed to be distributed according to the betweenness centrality of links (hereafter we mark it as C-scheme), it is found that synchronizability reaches its maximum only when the distributions of these two quantities match. In both cases, the couplings are directed and in general they are not balanced, i.e., one direction weights over another direction.

As more and more evidences point to the important roles of gradient couplings, an interesting question is: *which kind of gradient will be more efficient in improving the network synchronizability?* The answer relies on two parallel investigations: how to set the gradient direction and how to distribute the gradient weight. In setting the gradient direction, an intuitive method is to let the gradient start from the larger degree node and point to the smaller one [18, 19, 20]. But analysis of non-diagonalizable networks suggest that this setting can be arbitrary, given no loops in the gradient network [21]. In distributing the gradient weight, the answer is even diverse and confusing, the proposed methods range from the uniform weight distribution [19] (hereafter we marked it as H-scheme) to the distributions based on local information of node degree [18, 21] and on global information of link betweenness [20]. Therefore the global picture of the function of gradient/asymmetric coupling is still not clear and further study is needed.

In this work, we will investigate the problem of network synchronization from the gradient network point of view and give a generic understanding to the functions of gradient/asymmetric couplings in improving network synchronizability. Under the framework of gradient network, not only the previous results about asymmetric coupling can be well unified, but also the picture of how to distribute the gradient direction and weight becomes clear and simple. For example, now it is straightforward to figure out the necessary conditions for heterogeneous network to have a stronger synchronizability than homogeneous network and, more importantly, to what extend the gradient could benefits the network synchronizability.

The rest of the paper will be organized as follows. In Section II we will introduce the idea of gradient network and some of its basic properties. Based on the previous experiences and the new understanding of gradient network, In Section III we will suggest the basic criteria for constructing optimal networks and propose a new coupling scheme as application. In Section IV

we will analysis the properties of the gradient network emerges in the new coupling scheme and, in Section V, show its efficiency in improving network synchronizability by the method of eigenvalue analysis. In Section VI we will discuss the multiple effects of gradient and show how the optimal gradient changes its value according to the network parameters. In Section VII we present the simulation results on coupled nonidentical chaotic oscillators. Finally we give the discussion and conclusion in Section VIII.

II. GRADIENT NETWORK

For any pair of connected nodes, there are actually two directed couplings. When the weights in two directions are equal, we say the couplings are symmetric, otherwise, the couplings are asymmetric. To highlight the role of asymmetric couplings, a natural way is to separate the mutual couplings into two parts: the symmetric part and the asymmetric part. While the symmetric part reflects the common strength that two nodes affect each other, the asymmetric part represents the dominant role that one node put to another node. Extend this kind of separation to the whole network we will find that the original asymmetrically coupled network can be separated into two subgraphs: *one undirected symmetric network and one directed gradient network.* Both two subgraphs have the same topology as the original network, but the coupling direction and weight have been changed, as schematically shown in Fig. 1. Specially, the gradient network consists of only directed links. While synchronization of symmetric network has been well explored in previous studies, the separation of gradient network from the asymmetric network could simplify the problem and make the analysis easier. Now our attention solely turns to the study of gradient network and investigate its effects on network synchronization.

Please note that the so formed gradient network is different to those conventional ones. Conventionally gradient network is defined as the collection of directed links pointing to each node from whichever of its near neighbors has the highest/lowest scalar [10]. For network of N nodes and average degree $< k >$, according to the conventional definition there will be only N directed links in the gradient network. While for the mutually coupled network, every link may have a asymmetric part and will contributes one gradient link. Thus the gradient network generated from coupled system actually has totally $< k > \times N/2$ directed links. By noting that the gradients pointing to one node may have different weights and the behavior of this node is mainly determined by the largest one (a specific parameter will be introduced later to balance the weight distribution), it is

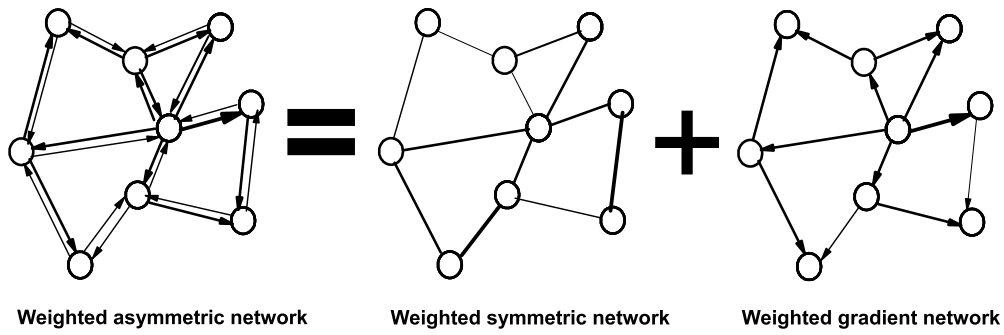

FIG. 1: The schematic diagram shows how to separate an weighted asymmetric network into a weighted symmetric network and a weighted gradient network.

reasonable to reduce the gradient network from $< k > \times N/2$ links to N links by only keeping the largest gradient pointing to each node. In other words, for gradient network of heterogeneous weight distribution, its main feature can be qualitatively captured by the reduced gradient network consisting of only the important gradients. With this simplification, the gradient network generated from coupled system will consist with those conventional ones, therefore many results in previous studies can be directed used here [10, 11]. Hereafter we will only analyze the properties and functions of this kind of reduced gradient network, while the correctness of this simplification will be verified later by eigenvalue analysis and direct simulations.

To concrete the idea further, we describe in following the conventional way of constructing gradient network [10]. Consider complex network $G = G(V, E)$ which has N nodes $V = \{1, ..., N\}$ and L links $E = \{1, ..., L\}$. The set of edges E is specified by a adjacency matrix $A = \{a_{i,j}\}$, $a_{i,j} = 1$ if i and j are connected, otherwise $a_{i,j} = 0$, while $a_{i,i} = 0$. For a given node i of degree k_i, the set of its neighbors is denoted by $V_i = \{j \in V \mid a_{i,j} = 1\}$. Consider also a scalar field $h = \{h_1, ..., h_N\}$ defined on the set of nodes V, therefore each node has a scalar value h_i associated to it. For node i of degree k_i, the gradient is defined as the directed link pointing to i from whichever of its k_i neighbors has the highest scalar. If the neighbors have the same scalar or if the node has the same scalar as all its neighbors, the selection will be random. Gradient network is just constructed as the collection of all these directed links. By this method, the generated gradient network consists of N nodes and N links. Except the node of the largest scalar

$h_l = \max\{h_1, ..., h_N\}$, which receives gradient from a node of smaller scalar than itself, all other nodes only receive gradient from nodes of higher scalar than themselves. Therefore, except the 2-node loop formed by the largest scalar node l and one of its neighbor node j, all the other links formed a tree structure with nodes l and j at the root. It has been shown that, for homogeneous network of random scalar distribution, the degree distribution of gradient network follows a power law scaling, $P(k) \sim k^{-\varsigma}$, with $\varsigma \approx -1$ [10].

Comparing to those conventional gradient networks, the gradient network generated from coupled system possesses some new features. Firstly, the node scalar usually are not of random distribution, it has a close relation to the properties of the node. Secondly, the gradient are weighted. Gradient between nodes of larger scalar difference usually assumes a different weight to that of smaller scalar difference. Finally, for sparsely connected networks we need to consider the problem of network breaking, i.e. the degeneration problem [10]. All these new features extend the conventional concept on gradient network and arise new problems for investigation. Our mission is just to characterize this kind of new gradient network and explore its functions in steering the collective behavior of coupled networks.

III. CONSTRUCTING THE COUPLING MATRIX

Gradient, also known as bias, has been employed in nonlinear studies for many years and proven to be a powerful technique in many fields such as chaos control and chaos synchronization. For example, for coupled oscillators on lattice it has been shown that the increase of gradient could greatly enhance the synchronizability of the system [22]. The asymmetric coupling schemes proposed in Refs. [18, 19, 20] suggest that gradient can be used to complex networks as well. Gradient has been also employed in turbulence control, where the introduction of gradient could significantly improve the control efficiency [23]. While different study proposes different scheme in setting the gradient, these schemes are nonetheless not unified. In particular, for complex networks, people are still not clear of the proper way of gradient configuration, i.e. how to set the gradient direction and weight. Now, from the view point of gradient network, these studies can be well unified and the principles for constructing optimal network can be vividly portrayed.

By reviewing the previous studies, we can get some important clues in constructing the matrix. (1) Based on the recent findings that 'hubs' are firstly synchronized than 'nonhubs' and the firstly synchronized 'hubs' act as the "core" in pattern formation of complex networks [24, 25], it seems

plausible to build mutual links between 'hubs' in the first place. In other words, the gradient between the hubs, if they are directly connected, should be small. By this setting we wish to build an efficient channel between the hubs and to form the synchronous core quickly. (2) To guarantee that the synchronous manifold of the "core" can be efficiently propagated to the 'nonhubs' while keeping the stability of the 'core' itself [19], it is reasonable to set the gradient start from 'hubs' and point to 'nonhubs'. According to this requirement we can set the gradient direction. (3) In practical systems the gradients pointing to one node generally have different weight, neighbors of higher scalar may generate larger gradient than those of lower scalar. To distinguish this difference, it is reasonable to set the gradient weight being proportional to the scalar potential. Setting in this way, the behavior of each node will more likely to follow its larger degree neighbors. Therefore, in determining the final synchronous state of the whole network, the high degree nodes have a larger contribution than the smaller degree nodes. (4) For most practical systems, each node has only limited information of the whole network, e.g. the degree information of itself or its neighbors. Which means that, in constructing the coupling matrix, the convenient method should employ only the local network information [18, 19].

Based on the above understandings, we propose a new coupling scheme for network synchronization. In particular, we consider networks of coupled chaotic oscillators following equations

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j=1}^{N} G_{i,j} \mathbf{H}[\mathbf{x}_j],\ i = 1, ...N, \tag{1}$$

where $\mathbf{F}(\mathbf{x}_i)$ governs the local dynamics of uncoupled node i, $\mathbf{H}[\mathbf{x}]$ is a linear coupling function and ε is the coupling strength. $G_{i,j}$ is a zero rowsum coupling matrix with off diagonal entries read

$$G_{i,j} = \frac{A_{i,j} k_j^{\beta}}{\sum_{j=1}^{N} L_{i,j} k_j^{\beta}},\ i, j = 1, ...N, \tag{2}$$

with A the adjacency matrix defined in Sec. II, k_j denotes the degree of node j, and β is a tunable parameter which will be used to adjust the gradient weight distribution. The diagonal entries are unit $G_{i,i} = 1$.

IV. THE PROPERTIES OF GRADIENT NETWORK

We first investigate the properties of the gradient network formed by Eq. 2. For a pair of connected nodes i and j, denote k_ι (k_j) the degree of node i (j) and V_i (V_j) the set of its neighbors,

then the gradient from i to j reads

$$\Delta G_{j,i} = G_{j,i} - G_{i,j} = \frac{k_i^\beta}{\sum_{l\in V_j} k_l^\beta} - \frac{k_j^\beta}{\sum_{l'\in V_i} k_{l'}^\beta} = \frac{1}{C_{i,j}}[k_i^\beta \sum_{l'\in V_i} k_{l'}^\beta - k_j^\beta \sum_{l\in V_j} k_l^\beta], \tag{3}$$

with $C_{i,j} = \sum_{l\in V_j}\sum_{l'\in V_i} k_l^\beta k_{l'}^\beta$. With the language of gradient network, it is straightforward to define the node scalar as follows

$$h_i = k_i^\beta \sum_{l\in V_i} k_l^\beta. \tag{4}$$

This scalar will be used to determine the gradient direction, i.e. from higher scalar to lower scalar; while the gradient weight is determined by the scalar potential $\Delta h_{i,j} = h_i - h_j$ and $C_{i,j}$, which depend on the degree situation of the node and its the neighbors. For SFN generated via the BA model [2], there is no degree correlation between the nodes [27], i.e. the chance to find a large degree neighbor or a small degree neighbor is the same for any given node. Specifically, we have for this kind of network the relation $\sum_{l\in V_i} k_l^\beta \sim k_i$. Substitute this relation into Eq. 4 we have

$$h_i \sim k_i^{1+\beta}. \tag{5}$$

It can be found that, for positive value of β, larger degree node generally possesses a larger value of h_i and the gradient is started from the larger degree node and point to the smaller degree node, while the contrary situation occurs when $\beta < 0$. (Note that with a slim chance the gradient may flow from smaller degree node to larger degree node. This situation happens when a smaller degree nodes has very large degree neighbors. However, this chance is very small and, statistically, the relation Eq. 5 is still valid, specially for network of dense connections.) Therefore, we can control the gradient direction by changing the sign of β, while control its weight, which is proportional to the scalar difference as shown in Eq. 3, by its absolute value $|\beta|$.

The constructed gradient network consists of N nodes and $< k > \times N/2$ directed links. To simplify the analysis, we assume $\beta \to \infty$ and discuss the reduced gradient network which consists of only N links. As we discussed in Sec. II, this reduction will not affect the main results we get for gradient network, but will make the analysis much easier. Now we argue that the formed gradient network is of forest structure (for obvious reason, we ignore the 2-node loop formed by the largest degree node and one of its neighbors) and its out-degree follows a power law degree distribution. To prove the forest structure, assume that on the contrary, there is a closed path $\Phi = \{\Delta G_{2,1}, \Delta G_{3,2}, ..., \Delta G_{l,l-1}\}, l \geq 3$ made up only of directed edges selected from the gradient network. Let j be the node on this path for which $h_j = min\{h_1, h_2, ..., h_m\}$. Node j has exactly

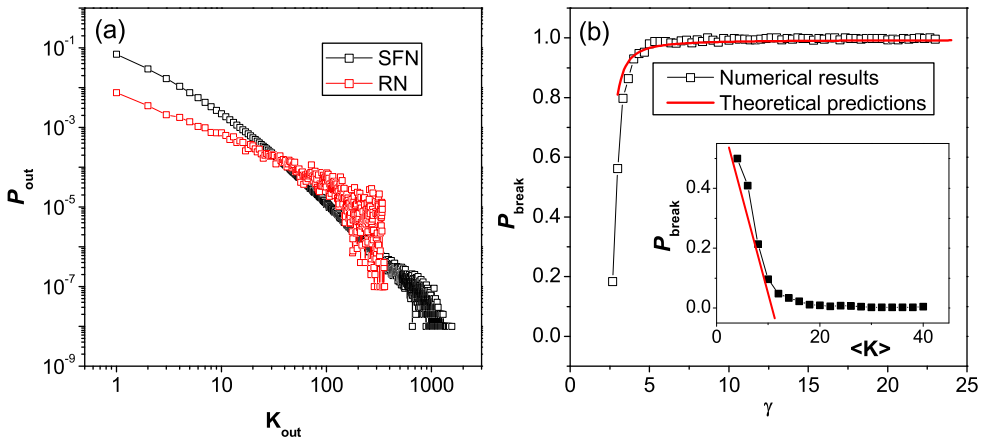

FIG. 2: (Color online). For network of $N = 2^{10}$ nodes and mean degree $< k >= 6$. (a) The out-degree distribution of the gradient networks generated on SFN and RN substrates. Both distributions follow power-law scaling with $\varsigma \approx -2.8$ for SFN substrate and $\varsigma \approx -1$ for RN substrate. (b) The probability of network breaking as a function of degree heterogeneity γ. Inset plots the breaking probability as a function of the mean degree $< k >$. It is found that network tends to be broken as γ increases or $< k >$ decreases. Each data is an average result over 100 realizations.

two neighbors on Φ, nodes $j \pm 1$, but only one gradient, $\Delta G_{j,j+1}$ or $\Delta G_{j,j-1}$, pointing to j. Since $h_j < h_{j\pm1}$, both of the neighbors $j \pm 1$ will have their gradient edges pointing into j. Since there are two edges, $(j-1, j)$ and $(j, j+1)$, and only one gradient edge will be accepted by j, one of the edges must not be a gradient edge, and thus the loop is not closed, in contradiction with the assumption that there is a loop with only gradient edges.

We go on to estimate the out-degree distribution. For node i which has degree k_i, assume node $l \in V_i$ is one of its neighbors, then the probability for l to receive gradient from i is determined by two elements: the probability of $h_i > h_l$ and the probability that h_i has the largest scalar among the neighbors of l, i.e. $h_i = \max\{h_j,\, j \in V_l\}$. Due to the zero degree correlation, the chance to find a connection between i and l is $p_1 = k_i/(N-1)$; according to Eq. 5, the probability for $h_i > h_l$ equals that of $k_i > k_l$ (here we discuss only the case of $\beta > 0$, while the analysis for $\beta < 0$ is the similar), which is $p_2 = \int_{k_{\min}}^{k_i} P(k)dk$; the probability that h_i is the largest scalar among the neighbors of l is $p_3 == 1 - (k_l - 1)p_3' = 1 - (k_l - 1)\int_{k_i}^{k_{\max}} P(k)dk$, where p_3' represents the probability that one link of l connects to some node of larger degree to i. Therefore, the probability

for l to contribute an out-degree to i is

$$p_l = p_1 p_2 p_3 = k_i/(N-1) \cdot \int_{k_{\min}}^{k_i} P(k)dk \cdot \left[1 - (k_l - 1)\int_{k_i}^{k_{\max}} P(k)dk\right]. \tag{6}$$

According to the network growth algorithm [2], we have $k_{\max} \approx m \times N^{\frac{1}{\gamma-1}}$ and $k_{\min} = m$ (m is the number of links associated to the new node in the BA growth model). Noting that the contribution of out-degree may come from any node of degree smaller than k_i, substituting the relation $P(k) = Ck^{-\gamma} = (\gamma-1)m^{\gamma-1}k^{-\gamma}$ into Eq. 6, we get the probability for each link of i to become an out-degree link is $\int_{k_{\min}}^{k_i} p_l N P(l)dl$. Finally, the totoal number of out-degree for node i is

$$\begin{aligned} k_{out} &= k_i \cdot \int_{k_{\min}}^{k_i} p_l N P(l)dl \approx \int_m^{k_i} \left\{k_i \cdot \int_m^{k_i} P(k)dk \cdot \left[1 - (l-1)\int_{k_i}^{k_{\max}} P(k)dk\right]\right\} Cl^{-\gamma}dl \\ &= C^2 k_i \cdot \int_m^{k_i} \left\{k_i \cdot \int_m^{k_i} k^{-\gamma}dk \cdot \left[1 - C(l-1)\int_{k_i}^{k_{\max}} k^{-\gamma}dk\right]\right\} l^{-\gamma}dl \end{aligned} \tag{7}$$

Denoting $\int_m^{k_i} k^{-\gamma}dk = a$ and $\int_{k_i}^{k_{\max}} k^{-\gamma}dk = b$, we have

$$\begin{aligned} k_{out} &= C^2 k_i \cdot \int_m^{k_i} \{k_i \cdot a \cdot [1 - C(l-1)b]\} l^{-\gamma}dl \\ &= aC^2 k_i \cdot \left\{(1+Cb)a - Cb\frac{m^{2-\gamma} - k_i^{2-\gamma}}{\gamma - 2}\right\} \end{aligned} \tag{8}$$

For $k_i \lesssim k_{\max}$, $Ca \approx 1$ and $Cb \approx 0$, we have

$$k_{out} \approx k_i \tag{9}$$

Therefore the out-degree distribution of the gradient network is the same to the degree distribution of SFN, which is

$$p_{out} \sim P(k) \sim k_i^{-\gamma}. \tag{10}$$

The exponent we get from numerical simulation matches this prediction very well, as shown in Fig. 2(a) where the fitted exponent is $\varsigma \approx -2.8$. Following the similar deduction, we can also proof that the out-degree distribution of random network also follows a power law distribution, but with exponent $\varsigma = -1$. This relation is verified by numerical simulation again and the result is plotted in Fig. 2(a) by another curve.

Using a similar reasoning as for the forest structure, we can prove that there is no continuous path which connects two local maxima of the scalar field h. This means that for each tree of the gradient network there is only one local maximum scalar, and it is the only node which forms

a 2-node loop with one of its neighbors. As a consequence, the number of trees in the forest equals the number of local maxima of the scalar field h. In other world, if there are more than one tree appear in the forest, the gradient network will be broken into disconnected small graphs. This kind of breaking phenomenon is crucial to network synchronization, since once the network is disconnected, it will never be synchronized whatever how large the coupling strength is. Therefore, to fully explore the function of gradient, we also need to estimate the breaking risk it may induce. Specifically, we want to know which kind of network could bear a larger gradient. In following we argue that *densely connected heterogeneous networks are more capable of larger gradient than sparsely connected homogeneous networks*. For SFN of degree heterogeneity γ, the largest node has about $k_{\max} \approx \frac{<k>}{2} N^{\frac{1}{\gamma-1}}$ links. Therefore the chance for the two leading nodes to be directly linked is about $P_{con} \approx 2k_{\max}/N$. In other words, the chance for them to be indirectly linked will be $P_{break} = 1 - P_{con}$. As the forest will broken into at least two trees when the two leading nodes are not directly connected, we have the network breaking probability

$$P_{break} = 1 - 2k_{\max}/N \approx 1 - < k > N^{\frac{2-\gamma}{\gamma-1}}. \tag{11}$$

This is only an estimation of the breaking probability, since the indirect connection of other relatively larger degree nodes, instead of the two largest ones, may also break down the forest in some cases. Such chance, however, is very small comparing to P_{break} and can be neglected, especially for networks of larger γ. (To construct networks of variable γ, we adopt the model proposed in Ref. [28], i.e. set a tunable parameter B to each node and adjust the preferential attachment function to be $p \sim (k_i + B)/\sum_j (k_j + B)$. The scaling exponent γ is then given by $\gamma = 3 + B/m$, with m the number of new links associated to each new node in network growth.) Numerical results on the breaking probability as functions of γ and $< k >$ are plotted in Fig. 2(b). It can be found that the predictions fit the numerical results reasonably well, especially in regions of larger γ and smaller $< k >$.

V. SYNCHRONIZABILITY ANALYSIS

We now analyze the function of gradient network to network synchronization. The synchronizability of coupled networks can be evaluated by the method of master stability function (MSF) [29, 30], if the eigenvalues are reals, or by the method of eigenvalue analysis [22], if the eigenvalues are complex values. These methods tell us that the problem of synchronizability can be

divided into two separating issues: the stability of the single dynamics $\mathbf{F}(\mathbf{x})$ and the distribution of eigenvalues of the coupling matrix G. For most systems, the single dynamics is stable within a certain range in the parameter space, $\sigma \in [\sigma_1, \sigma_2]$. The network is synchronizable iff all the eigenvalues except the one $\lambda_1 = 0$, which corresponding to the synchronous manifold, can be contained within this range after a linear scaling, i.e. $\lambda_N/\lambda_2 \leqslant \varepsilon\sigma_2/\sigma_1$, with λ_N the largest and λ_2 the smallest positive eigenvalues, respectively. In other words, the quantity of synchronizability can be described by the eigenratio $R = \lambda_N/\lambda_2$, with a smaller R represents a stronger synchronizability. Meanwhile, when network is synchronized, larger λ_2 usually means smaller coupling cost since $\varepsilon > \sigma_1/\lambda_2$.

The eigenvalues of asymmetric matrix G are usually complex values [19, 22], but for the coupling matrix G constructed in Eq. 2, they are reals. Noticing that the coupling matrix can be written as $G = QLD^{\beta}$, with $D = diag\{k_1, k_2, ...k_N\}$ the diagonal matrix of degrees and $Q = diag\{1/\sum_j L_{1,j}k_j^{\beta}, ...1/\sum_j L_{N,j}k_j^{\beta}\}$ the normalization factors for rows of G. From the following identity

$$\det(QLD^{\beta} - \lambda I) = \det(Q^{1/2}D^{\beta/2}LD^{\beta/2}Q^{1/2} - \lambda I) \tag{12}$$

we can find that the eigenvalues of the asymmetric matrix G are the same as that obtained from the symmetric matrix $H = Q^{1/2}D^{\beta/2}LD^{\beta/2}Q^{1/2}$, which are real and nonnegative values.

From the viewpoint of gradient network, the previous coupling schemes [18, 19, 20] can be well unified and the role of gradient can be easily identified. In M-scheme the gradient is generated according to the degree difference and adjusted via parameter β_M, i.e. $h_i = 1/k_i^{\beta}$ and $\Delta G_{j,i} = (k_i^{\beta} - k_j^{\beta})/(k_i^{\beta}k_j^{\beta})$. A negative value of β_M represents that the gradient flows from the smaller degree node to the larger degree node, while for positive β_M the gradient flows in the opposite direction. As reported In Ref. [18] and repeated in Fig. 3(a), the maximum synchronizability happens at $\beta_M \approx 1$. Since the gradient increases its weight as β_M increases from 0, it is of certain surprise to find that, instead of enhancement, larger gradient will suppress synchronizability when $\beta_M > 1$. Another intriguing observation is the sharp change of eigenratio R as β varies: $R \approx 2 \times 10^3$ at $\beta_M = -1$ while $R \approx 6$ at $\beta_M = 1$. While the optimization at $\beta_M \approx 1$ can be understood by the heterogeneous distribution of the coupling capacity among the nodes (at $\beta_M = 1$ the coupling capacity is the same for all nodes) and the decreased total coupling cost as β_M increases from the optimal value (as β_M increases from 1 the total coupling of the network will be decreased), the sharp change of R demonstrates the nontrivial effect of gradient played in network

synchronization. In Fig. 3(a) we also plotted the variations of the eigenratio as a function of the gradient for C-scheme, where gradient is generated by the betweenness difference between the connected nodes, and for H-scheme, where gradient is generated by the aging difference. For C-scheme, the gradient weight is adjusted by parameter β_C, with negative values represent that the gradient flows from node of smaller betweenness to node of larger betweenness, while the opposite happens for positive β_C. Again, the absolute value of β_C represents the weight of gradient. The behavior of R is quite similar to that of M-scheme, i.e. optimal configuration exists at around $\beta_C \approx 1$ while larger gradients suppress synchronizability, except that the variation of eigenratio is much slow than that of M-scheme. For H-scheme the gradient weight is adjusted via the parameter β_H, with negative values represent that the gradient flows from the 'older' (larger degree) node to the 'younger' (smaller degree) node, while opposite happens for positive β_H. Again, the absolute value of β_H represents the weight of gradient. Different to the former two schemes, it is found in Fig. 3(a) that for H-scheme the eigenratio R monotonically decreases as β_H decreases from 1 to -1, or, similarly, as the gradient from 'older' to 'younger' increases its weight. Noticing that $\beta_H = 0$ equals the situation $\beta_M = 1$ in M-scheme, it seems that for H-scheme the increase of gradient will always enhance synchronizability. However, from the view point of gradient network, the increase of gradient in H-scheme may induce the breaking problem which will suppress synchronization. Meanwhile, a 'younger' node receives gradients *equally* from all its 'older' neighbors despite their detail difference, this may confuse the target synchronous state to which the 'younger' node should follow [15]. Therefore the gradient functions are not fully explored in H-scheme.

As a comparison, we also plot in Fig. 3(a) the result of the new scheme described by Eq. 2. For $\beta < 0$, the gradient flows from smaller degree node to larger degree node and the opposite happens when $\beta > 0$. The weight of the gradient is adjusted by the absolute value of β. It can be found that, as β increases, the eigenratio R *monotonically* decreases from large values to small values. As we will show later, the smallest value that R can reach is only determined by the largest eigenvalue λ_N, which is almost constant for different coupling schemes. (In the extreme situation of $\beta \to \infty$, the coupling matrix takes the form of the reduced gradient network. There will be only two eigenvalues, λ_N and λ_2, while λ_2 equals 1.) When $\beta = 0$, we recover to the situations of $\beta_H = \beta_C = 0$ and $\beta_M = 1$ used in the previous schemes, respectively. Significantly, the maximum synchronizability at $\beta_H = -1$ can be achieved at around $\beta \approx 5$, while R can be further decreased as β increases in the new scheme. In this sense, we say that the new scheme is more efficient in

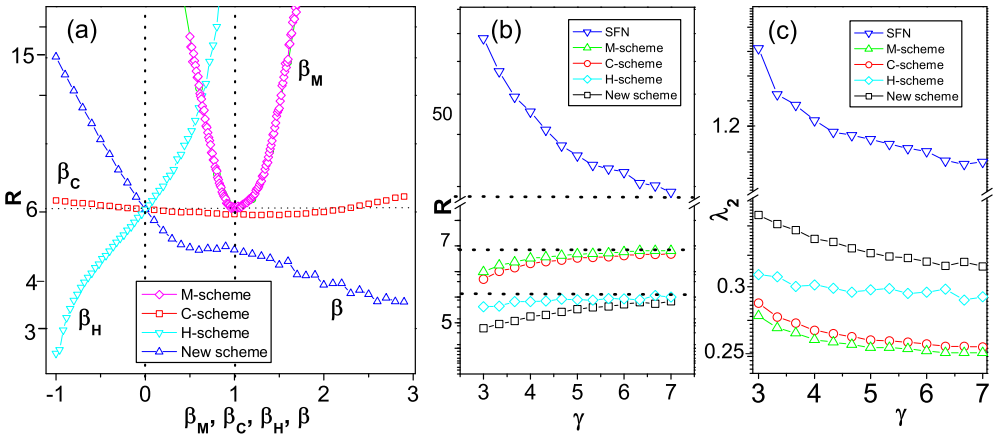

FIG. 3: (Color online). For the same SFNs as in Fig. 2. (a) The variations of eigenratio R as functions of the gradient parameters, β_M, β_C, β_H, and β, for different coupling schemes. (b) The eigenratio R as a function of degree heterogeneity γ for different coupling schemes. The three dashed lines represent the eigenratio of RN under the situation of, from top to bottom, without gradient, with M-scheme, and with the new scheme of Eq. 2, respectively. (c) The variation of the λ_2 as a function of heterogeneity for different schemes. Each data is averaged over 50 realizations.

employing gradient than the other schemes.

Noticing the fact that increasing degree heterogeneity could suppress synchronizability [15], we go on to compare the synchronizabilities of the different schemes as a function of heterogeneity. The variation of R as a function of γ is plotted in Fig. 2(b) together with four reference configurations: SFN and RN without gradient, RN of M-scheme, and RN of the new coupling scheme. To make the comparison fair, we adopt $\beta_M = 1$ and $\beta_C = 1$, where the maximum synchronizabilities are reached for the corresponding schemes. To make a fair comparison between the new scheme and the H-scheme, we adopt $\beta_H = -0.5$ and $\beta = 1.5$, since under this setting the total gradient is equal for these two schemes. It can be found that the new scheme has a clear advantage over all the other schemes. It is also found that, under the new scheme, networks of higher heterogeneity shows a much clear advantage over the homogeneous ones, while for the other schemes the advantage is relatively weak. Based on this observation, we say that the new scheme not only efficiently enhances the synchronizability of SFNs, as compared to their original configurations, but also makes SFNs prominently superior to RN. This finding provides a more stronger explanation to the paradox of network synchronization [18]. Another quantity used to

characterize synchronizability is λ_2, the second smallest eigenvalue. Since a larger value of λ_2 usually represents a smaller coupling cost in reaching the global synchronization, i.e. $\varepsilon > \sigma_1/\lambda_2$. The variation of λ_2 as a function of heterogeneity is plotted in Fig. 3(c), it is found that the new scheme has a larger value of λ_2 in comparison with the other schemes.

VI. UNDERSTANDING THE MULTIPLE EFFECTS OF GRADIENT

To explore the underlying mechanisms behind the new scheme, and also to manifest the basic principles we proposed for network construction in Sec. III, we go on to characterize the gradient network by other two quantities: the distribution of gradient weight and the distribution of eigenvalues. The gradient weight, as described by Eq. 3, is some value between $\Delta G_{\min} \approx 0$, where the connected nodes have the similar scalar, and $\Delta G_{\max} \approx 1$, where the link connect the largest degree and the smallest nodes. In Fig. 4(a) we plot the weight distributions of the gradient networks for all the coupling schemes. A clear difference is that the weight distribution of the new scheme has a long tail. According to Eq. 2, a larger $\Delta G_{i,j}$ represents a larger degree difference between the connected nodes. This is in accordance with the second principle in network construction, i.e. the gradient weight should be proportional to the the scalar potential. As a comparison, in other schemes the node receives gradients from it neighbors in a relatively mean fashion. For example, In H-scheme, the 'younger' node receives equal gradient from all the 'older' neighbors, while disregards the degree difference among them. The advantage of the new scheme is also reflected in the eigenspectrum. In Fig. 4(b) we plot the eigenvalue distributions for different coupling schemes. It can be found that, while the largest eigenvalue λ_N is similar for all the schemes, the new scheme is distinct with a larger value of λ_2 and an absolutely higher probability around $\lambda = 1$. This property makes the eigenvalues to be restricted within a tight region around $\lambda = 1$, and make the value of eigenratio R be quickly decreased as β increases. The extreme situation will be all the eigenvalues, except λ_N and λ_1, equal 1, where the maximum synchronizability is achieved and $R = \lambda_N$.

From Figs. 4(a) and (b), it seems that increasing gradient will monotonically increases the synchronizability. However, as shown in Fig. 2(b), too large gradients may suppress synchronizability, since gradient could induce the risk of network breaking. Now we show this phenomenon via eigenvalue analysis. In Fig. 4(c) we plot eigenratio R versus gradient β for networks of different heterogeneities. It is found that, as the heterogeneity decreases (i.e. increase the value of

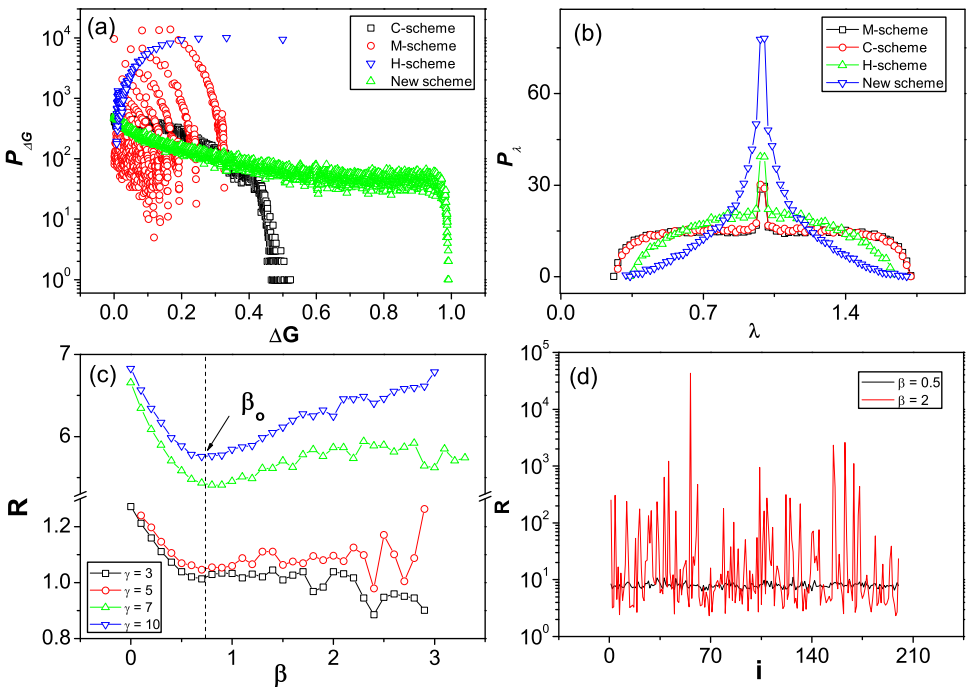

FIG. 4: (Color online). For the same SFN as in Fig. 2. Comparing (a) the distributions of gradient weight ΔG and (b) distributions of eigenvalues among the different coupling schemes. (c) For the new scheme, the variation of R as a function of β for networks of different heterogeneities. Optimal gradient happens around $\beta_o \approx 0.7$ when $\gamma = 10$. (d) The distribution of R as a function of the network realizations for different β at $\gamma = 10$.

γ), in the region of β, the gradient gradually changes its role from enhancing synchronizability to suppressing synchronizability. That is, for networks of lower heterogeneity and smaller mean degree, there exist an optimal gradient at β_o. When $\beta < \beta_o$, increasing gradient will enhance synchronizability, but when $\beta < \beta_o$, the opposite happens. The position where this transition appears depends on the values of γ and $< k >$, for densely connected SFN the value β_o will be significantly delayed. The breaking effect can be further understood from Fig. 4(d), where we plot the eigenratio R for a number of network realizations for fixed γ and $< k >$. It is found that, for smaller gradient $\beta < \beta_o$, R keeps on small values, representing that the network is still well connected. In this region, increase β will enhance the synchronizability; but for larger gradient $\beta > \beta_o$, R intermittently jumps to very large values, reflecting that the network is broken in some realizations. Therefore in this region increase β will suppress the synchronizability.

VII. NUMERICAL SIMULATIONS

We now provide the results of direct simulations. It was shown that, although the MSF method was proposed for complete synchronization of coupled identical systems, the eigenratio R could still provide a qualitative description for the collective behaviors, e.g. phase synchronization, of coupled nonidentical systems [18, 20, 24]. We employ SFN of nonidentical chaotic Rössler oscillators as the model. The dynamics of a singular oscillator reads $\mathbf{F}_i(\mathbf{x}_i) = [-\omega_i y_i - z_i, \omega_i x_i + 0.15 y_i, z_i(x_i - 8.5) + 0.4]$, with ω_i the natural frequency of oscillator i, which is randomly assigned in range $[0.9, 1.1]$. The coupling form is $\mathbf{H}(\mathbf{x}) = \mathbf{x}$. The degree of synchronization in this model can be characterized by monitoring the amplitude A of the mean field $X = \sum_{i=1}^{N} x_i/N$ [18, 24]. For small coupling strength, X oscillates irregularly and A is approximately zero, reflecting a smaller degree of synchronization; while X oscillates regularly and A increases sharply as coupling strength exceeds a critical value, reflecting a larger degree of synchronization. In Fig. 5(a) we plot the behavior of A as a function of the coupling strength ε for all the coupling schemes, it is found that the new scheme has a clear advantage over the other schemes even for smaller β, especially in the small the region of small couplings. To demonstrate the positive effect that gradient plays in the new scheme, we plot in Fig. 5(b) the behavior of A as a function of the gradient degree β. Again, as predicted by the eigenvalue analysis, A increases monotonically as β increases.

To show the breaking effect, we have carried the same simulations but for a homogeneous sparse network, i.e. $< k >= 4$ and $\gamma \to \infty$. As shown by the inset in Fig. 5(b), there is indeed a maximum around $\beta_o \approx 1$. Before this value, the increase of gradient will monotonically enhance the synchronizability, while after this value, increase gradient will suppress synchronizability. As we have analyzed, too large gradient induce the breaking effect and could suppress synchronization. However, in the region of $\beta > \beta_o$, nodes belong to the same tree are still synchronized and the network breaks into synchronized clusters, therefore it is expected that the decrease of A after β_o will be slow. Since this time although the network is not globally synchronized, strong coherence still exists within the clusters. The simulation results of Fig. 5(b) also testified our analysis that the network breaking effect is closely related to the mean degree $< k >$ and the degree heterogeneity γ, i.e. Eq. 11. Since in Fig. 5(b), where the substrate is densely connected SFN, we can not find the transition phenomenon, while for the sparsely connected RN network for the inset plot, we find the transition.

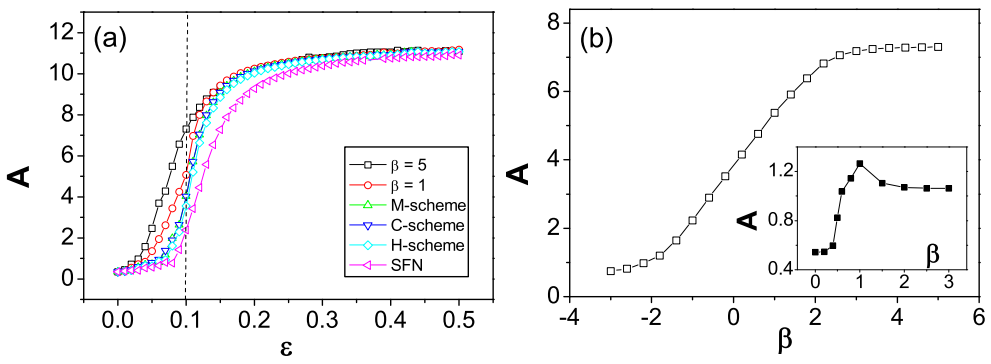

FIG. 5: (Color online). For SFNs of $N = 2^{10}$ nodes and mean degree $< k >= 10$, directed simulation of coupled nonidentical Rössler oscillators. (a) The mean field amplitudes as a function of coupling strength for different coupling schemes. The parameters are $\beta_M = 1$ for M-scheme, $\beta_C = 1$ for C-scheme, $\beta_H = -0.5$ for H-scheme, $\beta = 1.5$ and 5 for the new scheme. The total gradient in H-scheme when $\beta_H = -0.5$ and -1 equal that of the new scheme when $\beta = 1.5$ and 5, respectively. (b) For the new scheme, fixing $\varepsilon = 0.1$, the variation of mean field amplitude as a function of gradient. Each data is averaged over 10 network realizations. Inset is plotted for sparsely linked homogeneous network where each data is averaged over 1000 network realizations.

VIII. DISCUSSION AND CONCLUSION

Now we are able to answer the paradox described at the beginning of this paper: under what conditions that heterogeneous networks will be superior to homogeneous ones in synchronizability. The key point is gradient. By modifying the strength of the directed couplings, we can adjust the gradient direction and weight. To make the SFN outstand, the proper way to set the direction is making the gradient flow from the larger degree node and point to the smaller degree node; and a convenient method to set the gradient weight can be making it be proportional to the degree difference. If we set the direction and weight in the opposite ways, SFN will be much difficult to synchronize than RN. This finding thus indicates that, with proper gradient, both two kinds of heterogeneities, the heterogeneous degree distribution and heterogeneous weight distribution, can be used to improve network synchronizability instead of suppression.

The advantage that SFN superior to RN becomes even clear when considering the breaking effect. According to our finding, in achieving a stronger synchronizability, heterogeneous network can bear larger gradient than homogeneous network. The optimal gradient β_o, where the maximum

synchronizability is achieved, is closely related to the heterogeneity exponent γ and the mean degree $< k >$. The finding that sparse network can not bear too much gradient may have important implications to the function of many natural networks which have sparse links while gradient exists, e.g., the metabolic network where $< k > \approx 7.4$ and the protein network where $< k > \approx 2.39$ [3, 4]. Remarkably, because most technological (e.g. the Internet and WWW) and biological networks (e.g. the protein and neural networks) have the property of disassortativity [27], i.e. larger degree nodes tend to repulse from each other, the breaking effect will be more relevant in these systems.

Another advantage enjoyed by this new scheme is that, in constructing the coupling matrix, it only employs the local network information. More accurately, the degree information of the node itself and its neighbors. While schemes based on global network information, e.g. the betweenness centrality employed in C-scheme and the oriented tree employed in Ref. [21], are possible for small size networks, the constructions based on local information, e.g. the node degree employed in M-scheme and H-scheme, could be more efficient and practical in practical. We note that in the new scheme the setting of the gradient is determined by the information of the neighbors, instead of the node itself. This is one of the key points making the new scheme prominent. We also note that the proposed new scheme only provide one choice in employing the gradient, other methods which may involve complex construction rules and global network information, i.e. to design the gradient case by case depending on the specific topology structure, may further promote the synchronizability. But these methods may not as convenient as this new scheme.

In conclusion, we have employed gradient network to the problem of network synchronization. Under this framework, previous studies on asymmetric networks can be well unified and the principles for synchronizability enhancement become clear and systematic. Our studies not only indicates that, comparing to homogeneous networks, scale-free networks are the natural choice for synchronization, but also predict that, for practical networks of low heterogeneity and small mean degree, there should exist an optimal gradient β_o where the synchronizability is maximized. We expect this prediction to be verified by empirical findings in future.

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